External vs Mechanical loading.
surface and volume loads.

Strength, stiffness and Elastic Modulus

stiffness → geometry dependent, EXTRINSIC

Elastic Modulus → material property, INTRINSIC

Strength ↓ load bearing/ design stress

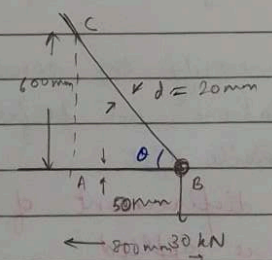
stiffness ↓ deformation under certain conditions (function of modulus and other factors)

Solid Mechanics ⟶ Mechanics + Material Science

* factor of safety ↪ overdesign
  ↓
  should be reduced
  * account for defects in the main eq^n s itself

→ Theoretical analyses and experimental observations have equal role here.

* Pyramids
  ↪ stresses remain same throughout

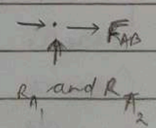


$\tan\theta = \frac{600}{800} = \frac{3}{4}$

$\sin\theta = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$

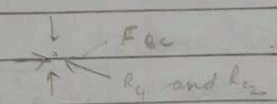
FBD at point A:-



$R_{A_1} e_1 + R_{A_2} e_2 + F_{AB} e_1 = 0$

$\therefore R_{A_1} + F_{AB} = 0$

At point C:-



$F_{BC}\left(\frac{-4}{5} e_1 + \frac{3}{5} e_2\right) = 0$

Rigid body (no internal)

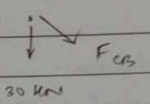
$A_x = 40$, $A_y = 0$, $C_x = -70$, $C_y = -30$

$F_{AB} = 40$ (C), $F_{BC} = 50$ (T)

$\left[\sigma_{AB} = \frac{F_{AB}}{A_{AB}}\right] \rightarrow 20.37$ MPa

<u>At point B:-</u>

$[\sigma_{BC} = 159.9 \text{ MPa}]$



* We do not consider any bending load while solving trusses.

Triangular design ensures that it remains non-collapsable. Hence we join diagonals to ultimately form triangles.

↳ no force members needed to improve stability.
(zero force)

* Externally loaded ~~to~~ joints speed up calculations.

* semi-circles also get bending loads, hence they are not used for heavy loading.

[COST VS STRENGTH]

↳ Strength
↳ deformation
↳ cost
↳ safety
↳ Weight

To verify the constitutional relations, we perform tensile test

Measure displacement of upper jaw and force applied.

↳ stress-strain curve

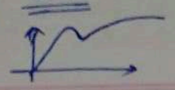
[NECKING / YIELD POINT]

δ^p and δ^e

Total deformation $\boxed{\delta = \delta^p + \delta^e}$

* Plastic deformation
(comes back parallel as modulus will remain same)

only steel has two yield points.
↳ due to presence of carbon

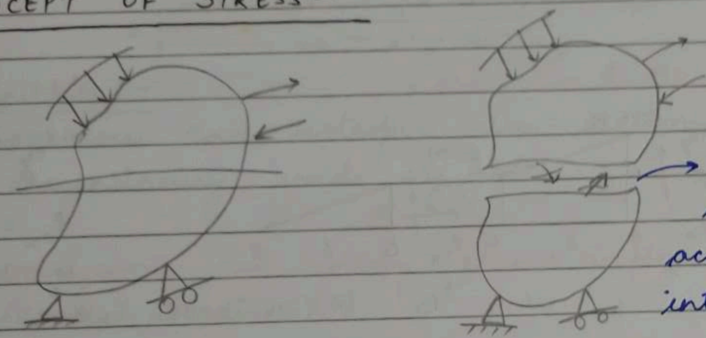


Energy corresponding to the permanent deformation is dissipated as heat.

Resilience and fracture energy

* Normal stress and shear stress
* Cauchy stress | Bernoulli stress (bending)

CONCEPT OF STRESS –



point wise not equal
⇑
→ expose the surface to account for internal forces.

$\frac{F}{A}$ gives average stress, but stress varies from point to point.

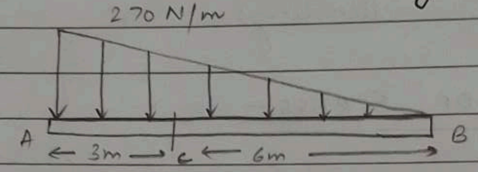
* Stress is a point function.

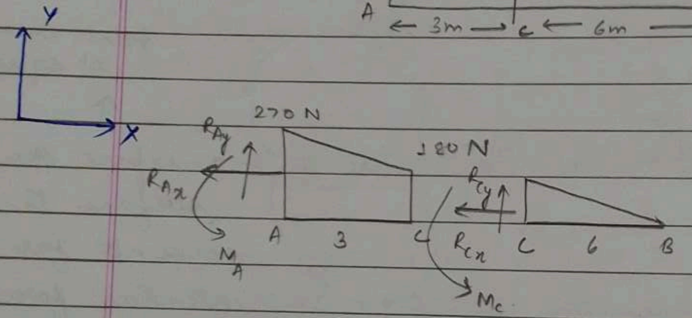
Pressure is always externally applied whereas stress is a result of internal resistance.

$\sigma_{xx}, \sigma_{xy}, \sigma_{xz}$ (depending on cross-section)

Normal force
Shear force
Torsional moment
Bending moment

Ques: Determine the internal loading at C.



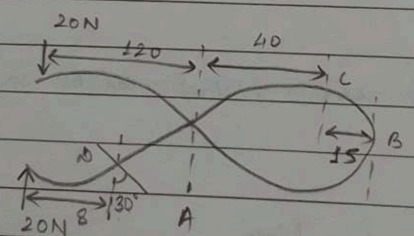


$R_{cy} = 540$

$R_{cx} = 0$

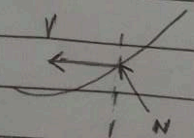
$M_c = 1080$

Ques:



Find reaction force at D and moment due to 20N force.

Total length = 175



Intensity of force

* That part of the body is likely to fail where the stresses are higher than the surrounding ones (point where crack starts).
  - ↳ Point wise distribution of forces.
  - ↳ Pass a cutting plane to predict the point — if this is safe, entire structure is safe

beam → concrete + metal
↳ stresses different for each

Assumptions:-

* Homogeneous and isotropic cross-section (→ properties independent of directions)
* No discontinuity : voids or holes

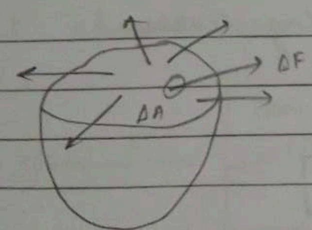
Homogeneity and isotropy do not imply each other.
(spatial coordinates)(θ)

Wood is non-homogeneous but isotropic [rings]
Crystals are homogeneous but ~~an~~ anisotropic.

(0th order tensor) (1st order tensor)

SCALARS, VECTORS, TENSORS

- SCALARS ↓ store one piece of information (magnitude)
- VECTORS ↓ store two pieces of information (magnitude and dirn)
- TENSORS: n^{th} order tensor stores $n+1$ pieces of information



$$\left[\lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \sigma \right]$$

TRACTION (not STRESS as it has come to the surface)

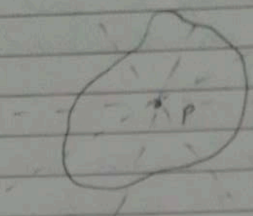
any force distribution over a surface

Timoshenko

$\boxed{\text{No. of components of a tensor} = 3^n}$

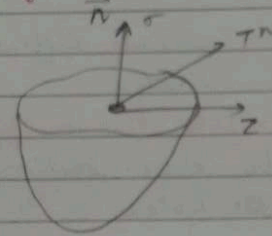
n = order of a tensor

w.r.t. a 3-D coordinate system



Infinitely many planes can pass through a point, with changes in the $\vec{n}$.
↳ unit normal

Totality of traction is the state of stress.



$$\left[{T^n}^2 = \sigma^2 + \tau^2 \right]$$

$$\left[{T^n}^2 = T_x^2 + T_y^2 + T_z^2 \right]$$

Consider a cubical element in the body. Its faces will have normals along the coordinate axes.

Order of tensor determines indices for description.

$\boxed{\sigma_{ij}}$

i = unit normal of the area on which the stress is defined

j = the direction in which the stress is acting.

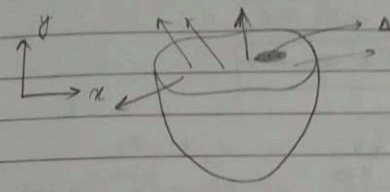
2, 3, 5, 9, 13, 21
↳ components based on symmetry

$\varepsilon_{ijk} \rightarrow$ alternating tensor

$C_{ijkl} \rightarrow$ stiffness

$$\left[\sigma_{ij} = \underset{\downarrow\, E}{C_{ijkl}}\, \varepsilon_{kl} \right]$$

Any bridge is statically indeterminate.



F = Force on area A

$\Delta F, \Delta T \Rightarrow$ force on area ΔA

$$\sigma_{avg} = \frac{F}{A}$$

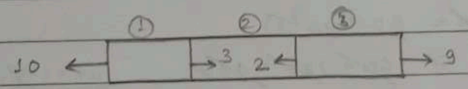
stress vector

$$\lim_{\Delta A \to 0} \frac{\Delta F (\text{or } \Delta T)}{\Delta A} = T \quad \text{(surface traction)}$$

(x, y, z) OR (x, y, z) OR $(1, 2, 3)$ OR (i, j, k)

$$\left[T = f(\sigma) \right]$$

Ques:



Which part has maximum stress?

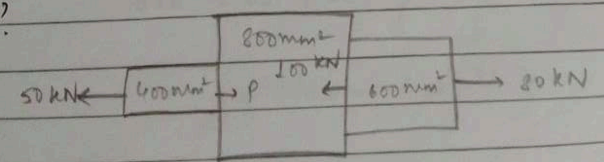
Step I Check whether it is static or dynamic.

$$\Sigma F_x = -10 + 3 - 2 + 9 = 0 \qquad \text{STATIC}$$

Step II Draw FBD for each part.

Max. force!

Ques: What should be max. P and σ_{max} for static case?



$\cancel{\Sigma F_x = -50}$ $\quad \Sigma F_x = 0$

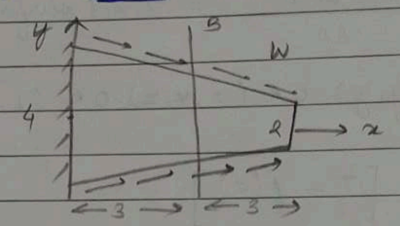
$$-50 + P - 100 + 80 = 0$$

or, $\boxed{P = 70 \text{ kN}}$

$$\sigma_1 = \frac{50}{400}, \quad \sigma_2 = \frac{20}{800}, \quad \sigma_3 = \frac{80}{600}$$

$\therefore \sigma_3$ is max.

Ques:-



force per unit length

$W = 60 + 40x$

$\sigma_B = ?$

AT POINT

$\cancel{\text{At } B,}$ $\cancel{W = 60 + 40(3)}$

$\cancel{= 60 + 120 = 180}$

$R_y = 0$

$M = 0$

$$A_B = \pi\left(\frac{3}{2}\right)^2 = \frac{9}{4} \times 3.14$$

$$\therefore \sigma_B = \frac{180 \times 2}{\frac{9}{4} \times 3.14} = \frac{4 \times \cancel{180}^{20} \times 2}{\cancel{9} \times 3.14}$$

$$= \frac{80 \times 2}{3.14} = \frac{160}{3.14}$$

force at the cross section is the load reaction at that cross-section.

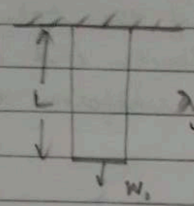
force at B : $\int W\,dx$

$$= \int_0^3 (60 + 40x)\,dx$$

$$= \left[60x + 20x^2\right]_0^3$$

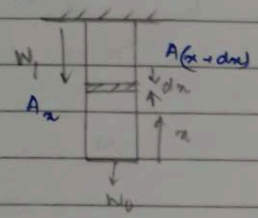
$$= 180 + 180 = 360$$

Ques.



$\lambda \rightarrow$ weight per unit length

What should be the shape of the bar so that each cross-section has uniform ~~stress~~ strength?



Consider a strip of width dx at a distance x from the base.

$$\sigma = \frac{W_0}{A}$$

$$\sigma = \frac{W_0 + W_1}{A_x}$$

$$= \frac{W_0 + W_1 + dW}{A(x+dx)}$$

$$f(A_x) = W_0 + W_1$$

$$f(A_x + dA_x) = W_0 + W_1 + (A_x\,dx)\lambda$$

$$f(dA_x) = (A_x\,dx)\lambda$$

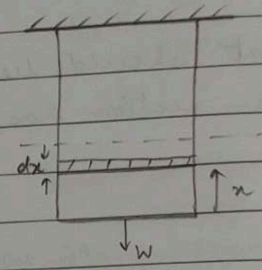
$$\int_A^{A_x} \frac{dA_x}{A_x} = \frac{\lambda}{f}\int dx$$

$$\Rightarrow \ln\frac{A_x}{A} = \frac{\lambda x}{f} \quad A_x$$

$$\neq \cancel{\frac{\lambda x}{f}} \quad \cancel{A}.$$

$$\therefore \left[f\,(dA_x) = A e^{\lambda x / f} \right]$$

⇒ Determine the shape of the bar having freely subjected to pull W at its lower end to have uniform strength f. Assume uniform density of the bar.



Taking a strip of width dx at a distance x from the bottom.

$$f = \frac{W}{A_0} = k$$

$$\therefore \frac{W + W_1}{A_x} = \frac{W + W_1 + \delta W}{A_{x+dx}}$$

Now, $\delta W = (\rho A_x dx) g$

$$f = \frac{W + W_1}{A_x} = \frac{W + W_1 + (\rho A_x dx) g}{A_{x+dx}}$$

$$f A_x = W + W_1$$

$$\therefore f = \frac{f A_x + (\rho A_x dx) g}{A_{x+dx}}$$

or, $f \cdot (A_{x+dx}) = A_x (f + \rho g \, dx)$

or, $f \left(\frac{A_{x+dx}}{A_x} \right) = f + \rho g \, dx$

$$A_{x+dx} = A_x + \delta A_x$$

$$\therefore f \left(\frac{A_x + \delta A_x}{A_x} \right) = f + \rho g \, dx$$

or, $f \left(1 + \frac{\delta A_x}{A_x} \right) = f + \rho g \, dx$

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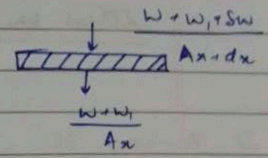
Rebound hammer measures stiffness.

* Bodies deform in such a way that its potential energy is minimum.
* Cracks occur to dissipate energy.

$$f = \frac{W + W_1}{A_x}$$



$$f = \frac{W + W_1 + \delta W}{A_{x} + dx}$$

$\therefore$ $f A_x = W + W_1$

$f(A_x + dx) = W + W_1 + dW$

$\therefore$ $f(A_x + dx - A_x) = dW$

or, $f(dA_x) = \rho A_x \, dx \, g$

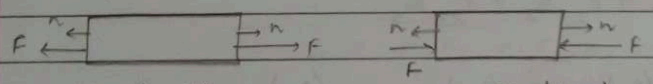
or, $\int f \frac{dA_x}{A_x} = \int \rho g \, dx$

or, $\left[f \log_e A_x \right]_{A_0}^{A_x} = \rho g x \Big|_0^x$

or, $f \log_e\left(\frac{A_x}{A_0}\right) = \rho g x$

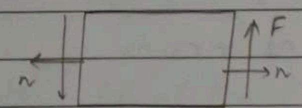
or, $\boxed{A_x = A_0 e^{\rho g x / f}}$

Area resisting the force should follow from the ~~complete~~ calculation.



tensile compressive

NORMAL



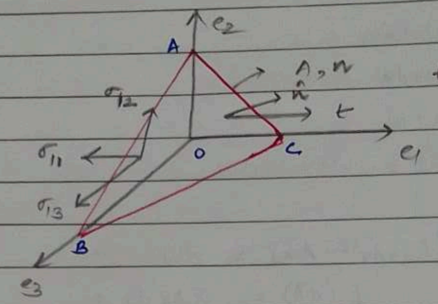
SHEAR

* single and double shear
  → preferred due to distribution (stress is less) in general

Imp. shear stress $\sigma = \dfrac{\text{Force}}{\text{Area which is resisting the force}}$

Stress on an arbitrary plane :-

If $A \to 0$, it converges to a point



$t = t_1, t_2, t_3$

$A_1 = A n_x$ (n_1), $A_2 = A n_y$ (n_2), $A_3 = A n_z$ (n_3)

$OAB \to \sigma_{11}, \sigma_{12}, \sigma_{13}$
$OBC \to \sigma_{22}, \sigma_{21}, \sigma_{23}$
$OCA \to \sigma_{33}, \sigma_{31}, \sigma_{32}$

for static equilibrium:

$\Sigma F_1 = 0$

$t_1 A = \sigma_{11} A_1 + \sigma_{21} A_2 + \sigma_{31} A_3$
$t_2 A = \sigma_{12} A_1 + \sigma_{22} A_2 + \sigma_{32} A_3$
$t_3 A = \sigma_{13} A_1 + \sigma_{23} A_2 + \sigma_{33} A_3$

$\therefore t_1 = \sigma_{11}\left(\dfrac{A_1}{A}\right) + \sigma_{21}\left(\dfrac{A_2}{A}\right) + \sigma_{31}\left(\dfrac{A_3}{A}\right)$

limiting case as $A \to 0$ $= \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$

Similarly, $t_2 = \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3$

$t_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3$

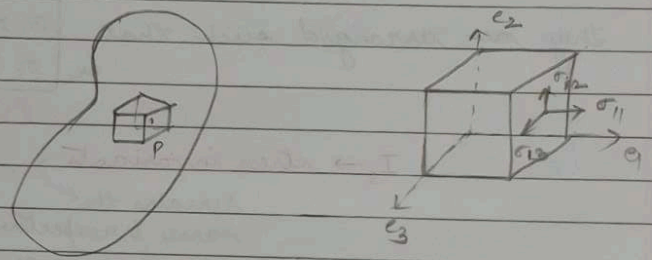
Same argument can be extended to include body forces as well.

Ductility and shear stress.
Brittleness and normal stress. }

$$\boxed{t_i = \sigma_{ji} n_j}$$

To get three components of traction, we need 9 components of stress.

$$\left\{ \begin{aligned} \sigma_N &= t \cdot n \\ z^2 &= t \cdot t - \sigma_N^2 \end{aligned} \right\}$$



We need to find the point where fracture starts. So we want to simplify it down to just 3 components.
This we achieve using EIGEN VALUES AND EIGEN VECTORS.

$$t_1 = \overset{(\sigma_{11}-\lambda)}{\sigma_{11}} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3$$
$$t_2 = \sigma_{12} n_1 + \overset{(\sigma_{22}-\lambda)}{\sigma_{22}} n_2 + \sigma_{32} n_3$$
$$t_3 = \sigma_{13} n_1 + \sigma_{23} n_2 + \overset{(\sigma_{33}-\lambda)}{\sigma_{33}} n_3$$

$$\begin{bmatrix} \sigma_{11}-\lambda & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22}-\lambda & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33}-\lambda \end{bmatrix} \rightarrow \lambda^3 + f_1\lambda^2 + f_2\lambda + f_3 = 0$$

3 roots
$\Rightarrow \lambda_1, \lambda_2, \lambda_3$

$\sigma_{11}', \sigma_{22}', \sigma_{33}' \longleftarrow \sigma_1, \sigma_2, \sigma_3$

acting on the eigen vector coordinate system
$\Rightarrow$ no components

* Normal to a principal plane is called principal ~~plane~~ stress axis.

Brice
Page: 16
Date: 17/08/22

let $\boxed{\lambda^3 - I_1(\lambda^2) + I_2(\lambda) - I_3 = 0}$ be the characteristic eqⁿ of the stress tensor matrix.

$\therefore I_1 = tr(\sigma)$

$I_2 = \frac{1}{2}\left[(tr\,\sigma)^2 - tr(\sigma^2)\right]$

$I_3 = det(\sigma)$

They are arranged such that $\lambda_1 > \lambda_2 > \lambda_3$
or, $\sigma_1 > \sigma_2 > \sigma_3$ → Principal stress

I_s → stress invariants → remains the same irrespective of the change in coordinate system

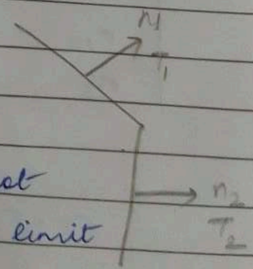
* Planes on which principal stresses act are called principal planes.

* Eigen vectors represent the principal directions.
↳ should satisfy $\boxed{n_x^2 + n_y^2 + n_z^2 = 1}$

Proof for orthogonality of principal planes:-

Using PROJECTION THEOREM,

$$\boxed{T_1 \cdot n_2 = T_2 \cdot n_1}$$



⇒ If n_1 and n_2^0 define two points (not necessarily orthogonal but in the limit passing through the same point).

Page: Date: / /

EQUILIBRIUM EQUATIONS-

* Can stress at the neighbourhood of point in a material take any value?

$$\Sigma_i F_x = 0$$

can be derived from the potential energy eqn as well.

$$\left(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1\right) \Delta x_2 \Delta x_3 + \left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2\right) \Delta x_1 \Delta x_3 + \left(\sigma_{31} + \frac{\partial \sigma_{31}}{\partial x_3} dx_3\right) \Delta x_2 \Delta x_1 - \sigma_{11}(\Delta x_2 \Delta x_3) - \sigma_{21}(\Delta x_1 \Delta x_3) - \sigma_{31}(\Delta x_2 \Delta x_1) = 0$$

or, $$\left(\frac{\partial \sigma_{11}}{\partial x_1}\right) dx_1 \Delta x_2 \Delta x_3 + \left(\frac{\partial \sigma_{21}}{\partial x_2}\right) dx_2 \Delta x_1 \Delta x_3 + \left(\frac{\partial \sigma_{31}}{\partial x_3}\right) dx_3 \Delta x_1 \Delta x_2 = 0$$

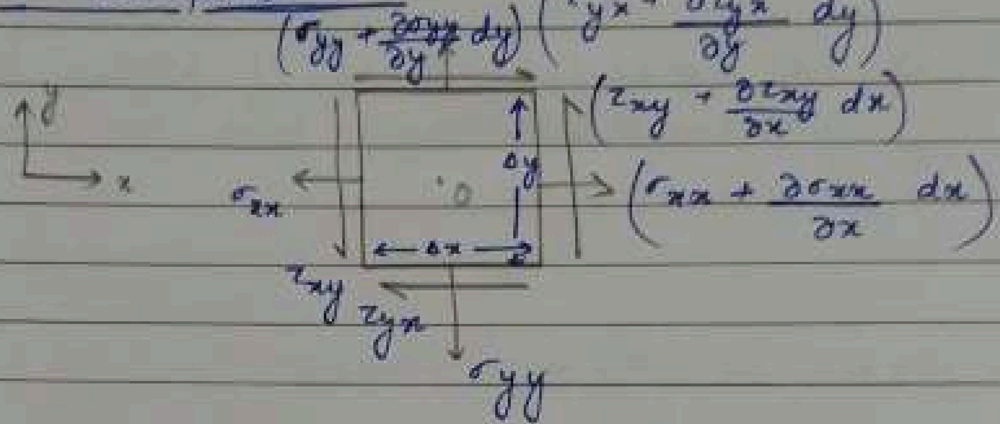
Taking the limit as $\Delta x_1, \Delta x_2, \Delta x_3 \to 0$, and dividing by $\Delta x_1 \Delta x_2 \Delta x_3$

$$\Rightarrow \left[\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} = 0\right]$$

If body force per unit volume, γ_x is also included

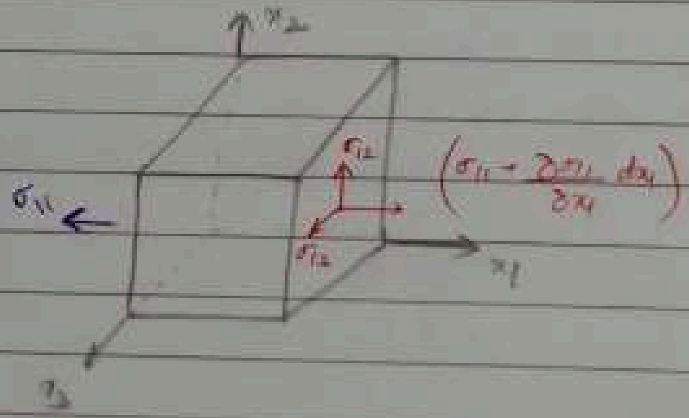
$$\left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \gamma_x = 0\right]$$

$\Sigma M_{\text{mid-point}} = 0$



$\Sigma M_o = 0 \quad \left[\left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx\right) \times dy\right] \frac{dx}{2}$

$\Rightarrow$ Assume a cube in 3-D.



$\Sigma M_3 = 0$

$$\left(\sigma_{11} + \frac{\partial \sigma_{11}}{\partial x_1} dx_1\right) \times \Delta x_2 \Delta x_3 \times \frac{\Delta x_2}{2} - \left(\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_1} dx_1\right) \Delta x_2 \Delta x_3 \Delta x_1$$

$$- \left(\sigma_{11} \Delta x_2 \Delta x_3\right) \times \frac{\Delta x_2}{2} \quad -$$

$$\left(\sigma_{22} + \frac{\partial \sigma_{22}}{\partial x_2} dx_2\right) \Delta x_1 \Delta x_3 \frac{\Delta x_1}{2} + \left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2} dx_2\right) \Delta x_1 \Delta x_2 \Delta x_3$$

$$+ \sigma_{22} \Delta x_1 \Delta x_3 \frac{\Delta x_1}{2}$$

$\sum M_{x_3} = 0$

$$\left(\cancel{\sigma_{11}} + \frac{\partial \sigma_{11}}{\partial x_1}\Delta x_1\right)\Delta x_2 \Delta x_3 \frac{\Delta x_2}{2} - \left(\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_1}\Delta x_1\right)\Delta x_2 \Delta x_3 \Delta x_1$$

$$- \cancel{\sigma_{11}}(\Delta x_2 \Delta x_3)\frac{\Delta x_2}{2} - \left(\cancel{\sigma_{22}} + \frac{\partial \sigma_{22}}{\partial x_2}\Delta x_2\right)\Delta x_1 \Delta x_3 \frac{\Delta x_1}{2}$$

$$+ \left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2}\Delta x_2\right)\Delta x_1 \Delta x_3 \Delta x_3 + \cancel{\sigma_{22}}\Delta x_1 \Delta x_3 \frac{\Delta x_1}{2}$$

$$- \left(\cancel{\sigma_{32}} + \frac{\partial \sigma_{32}}{\partial x_3}\Delta x_3\right)\Delta x_1 \Delta x_2 \frac{\Delta x_1}{2} + \left(\cancel{\sigma_{31}} + \frac{\partial \sigma_{31}}{\partial x_3}\Delta x_3\right)\Delta x_1 \Delta x_2 \frac{\Delta x_2}{2}$$

$$+ \left(\cancel{\sigma_{32}}\Delta x_1 \Delta x_2 \frac{\Delta x_1}{2}\right) - \cancel{\sigma_{31}}\Delta x_1 \Delta x_2 \frac{\Delta x_2}{2} = 0$$

Dividing both sides by $\Delta x_1 \Delta x_2 \Delta x_3$

$$\Rightarrow \frac{\partial \sigma_{11}}{\partial x_1}\cancel{\frac{\Delta x_2}{2}}^{0} - \left(\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_1}\cancel{\Delta x_1}^{0}\right) - \frac{\partial \sigma_{22}}{\partial x_2}\cancel{\frac{\Delta x_1}{2}}^{0} +$$

$$\left(\sigma_{21} + \frac{\partial \sigma_{21}}{\partial x_2}\cancel{\Delta x_2}^{0}\right) - \frac{\partial \sigma_{32}}{\partial x_2}\cancel{\frac{\Delta x_1}{2}}^{0} + \frac{\partial \sigma_{31}}{\partial x_3}\cancel{\frac{\Delta x_2}{2}}^{0} = 0$$

as for a point $\Delta x_1, \Delta x_2, \Delta x_3 \to 0$

$$\cancel{\sigma_{12}} \; -\sigma_{12} + \sigma_{21} = 0$$

or $\boxed{\sigma_{12} = \sigma_{21}}$

Stress is symmetric for the coordinate axes.

$$\boxed{\sigma_{xy} = \sigma_{yx}}$$

By PROJECTION THEOREM,

$$\sigma_1 n_1 . n_2 = \sigma_2 n_2 . n_1$$

or $n_1 . n_2(\sigma_1 - \sigma_2) = 0$ $\quad [n_1 . n_2 = n_2 . n_1]$

$\Rightarrow \boxed{n_1 \perp n_2} \quad \forall \; \sigma_1, \sigma_2$

$$\left[c_{ijkl}' = a_{ip}\, a_{jq}\, a_{kr}\, a_{ls}\, c_{pqrs} \right]$$

Now, as $[\sigma_{xy} = \sigma_{yx}]$ $\quad \sigma_{12} = \sigma_{21}$

$\therefore$ Projection planes are orthogonal.

$$\therefore \left[\sigma^3 - I_1 \sigma^2 + I_2 \sigma - I_3 = 0 \right]$$

→ 3 real roots

$\sigma_1, \sigma_2, \sigma_3$ → principal stress eigen values
n_1, n_2, n_3 → principal value eigen vector.

Any plane ~~stress~~ where stress is along its normal is the principal plane.

$\sigma_1, \sigma_2, \sigma_3$ for 3 real roots → 3 principal dir^n [only real roots is considered]
$\sigma_1 = \sigma_2, \sigma_3$ for 1 real root → 1 principal dir^n (unique) → rest in the ⊥ plane
$\sigma_1 = \sigma_2 = \sigma_3$ for no real root → infinite principal dir^n
(hydrostatic state of stress)

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

No. of components = 9
No. of independent components = 6

$$\left\{ \sigma_{ij} \longrightarrow \sigma_{ip}^{\,p} \right\}$$

$$\boxed{v_i' = a_{ij} v_j}$$

$$\boxed{a_{ij} = \cos \theta_{ij}}$$

Any system can be translated and rotated (any comb^n) to a new one.

θ_{ij} = angle b/w new i and old j

⇒ Tensor is a mapping b/w two vectors.

$$\vec{F} = \sigma A \vec{n}$$

generally double.

We need a 2nd order tensor to map two vectors.

$$u = Tv$$

$$u_i = T_{ij} v_j$$

$$u_i' = T_{ij}' v_j'$$

Dummy index

$u_i' = a_{ij} u_j$

$v_p' = a_{pq} v_q$

$\Rightarrow \left[v_q = \cancel{a_{pq}}\, v_p' \right]$ a_{qp}

* free index and place of dummy index should remain the same.

$$u_i' = a_{ij} u_j$$
$$= a_{ij} T_{jp} v_p$$
$$= a_{ij} T_{jp} a_{qp} v_q'$$

These are components of the matrix, but not the matrix ~~itsef~~ itself.

or, $u_i' = a_{ij} a_{qp} T_{jp} v_q'$

$\boxed{\text{No. of free indices} = \text{order of tensor}}$

$$T_{ij}' = a_{ip} a_{jq} T_{pq} \qquad \left[\begin{aligned} & a_{ij} a_{qp} T_{jp} \\ &= a_{ip} a_{jq} T_{pq} \end{aligned} \right]$$

$$\therefore \left[\sigma_{ij}' = a_{ip} a_{jq} \sigma_{pq} \right]$$

While transforming to principal coordinates, matrix consists of $\{n_1, n_2, n_3\}$

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$$t_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 = \sigma_1 n_1$$

Similarly, $t_2 = \sigma_2 n_2$

$t_3 = \sigma_3 n_3$

$$\sigma_N = t \cdot n$$

$$\left[\sigma_N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \right] \text{ — (i)}$$

$$\left[\tau^2 = T^2 - \sigma_N^2 \right]$$

$$\left[n_1^2 + n_2^2 + n_3^2 = 1 \right] \text{ — (iii)} \qquad \left[\tau^2 + \sigma_N^2 = T^2 \right] \text{ — (ii)}$$

$$T^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_3^2 n_3^2$$

(ii) $- \sigma_2$(i)

$$= n_1^2(\sigma_1^2 - \sigma_1\sigma_2) + n_2^2(\sigma_2^2 - \sigma_2^2) + n_3^2(\sigma_3^2 - \sigma_2\sigma_3)$$

$$= n_1^2(\sigma_1^2 - \sigma_1\sigma_2) + n_3^2(\sigma_3^2 - \sigma_2\sigma_3)$$

$$= \tau^2 + \sigma_N^2 - \sigma_2\sigma_N$$

$$\therefore \left[n_3^2 = \frac{(\tau^2 + \sigma_N^2 - \sigma_2\sigma_N) - n_1^2(\sigma_1^2 - \sigma_1\sigma_2)}{\sigma_3^2 - \sigma_2\sigma_3} \right]$$

(ii) $- \sigma_3$(i)

$$n_1^2(\sigma_1^2 - \sigma_1\sigma_3) + n_2^2(\sigma_2^2 - \sigma_2\sigma_3) = \tau^2 + \sigma_N^2 - \sigma_3\sigma_N$$

$$\therefore \left[n_2^2 = \frac{(\tau^2 + \sigma_N^2 - \sigma_3\sigma_N) - n_1^2(\sigma_1^2 - \sigma_1\sigma_3)}{\sigma_2^2 - \sigma_2\sigma_3} \right]$$

$$\because n_1^2 + n_2^2 + n_3^2 = 1$$

$$\therefore n_1^2 + \left(\frac{\tau^2 + \sigma_N^2 - \sigma_3\sigma_N}{\sigma_2^2 - \sigma_2\sigma_3} \right) - n_1^2 \left(\frac{\sigma_1^2 - \sigma_1\sigma_3}{\sigma_2^2 - \sigma_2\sigma_3} \right)$$

$$+ \left(\frac{\tau^2 + \sigma_N^2 - \sigma_2\sigma_N}{\sigma_3^2 - \sigma_2\sigma_3} \right) - n_1^2 \left(\frac{\sigma_1^2 - \sigma_1\sigma_2}{\sigma_3^2 - \sigma_2\sigma_3} \right) = 1$$

or $$n_1^2 \left[1 - \left(\frac{\sigma_1^2 - \sigma_1\sigma_3}{\sigma_2^2 - \sigma_2\sigma_3} \right) - \left(\frac{\sigma_1^2 - \sigma_1\sigma_2}{\sigma_3^2 - \sigma_2\sigma_3} \right) \right] = 1 - \left(\frac{\tau^2 + \sigma_N^2 - \sigma_3\sigma_N}{\sigma_2^2 - \sigma_2\sigma_3} \right)$$

$$- \left(\frac{\tau^2 + \sigma_N^2 - \sigma_2\sigma_N}{\sigma_3^2 - \sigma_2\sigma_3} \right)$$

$$1 - \left(\frac{\sigma_1^2 - \sigma_1\sigma_3}{\sigma_2^2 - \sigma_2\sigma_3}\right) - \left(\frac{\sigma_1^2 - \sigma_1\sigma_2}{\sigma_3^2 - \sigma_2\sigma_3}\right) \star$$

$$= \frac{(\sigma_2^2 - \sigma_2\sigma_3)(\sigma_3^2 - \sigma_2\sigma_3) - (\sigma_1^2 - \sigma_1\sigma_3)(\sigma_3^2 - \sigma_2\sigma_3) - (\sigma_1^2 - \sigma_1\sigma_2)(\sigma_2^2 - \sigma_2\sigma_3)}{\sigma_2(\sigma_2 - \sigma_3)\,\sigma_3(\sigma_3 - \sigma_2)}$$

$$= \frac{\sigma_2\sigma_3(\sigma_2 - \sigma_3)(\sigma_3 - \sigma_2) - \sigma_1\sigma_3(\sigma_1 - \sigma_3)(\sigma_3 - \sigma_2) - \sigma_1\sigma_2(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)}{-\sigma_2\sigma_3(\sigma_2 - \sigma_3)^2}$$

$$= \frac{-\sigma_2\sigma_3(\sigma_2 - \sigma_3)^2 + \sigma_1\sigma_3(\sigma_1 - \sigma_3)(\sigma_2 - \sigma_3) - \sigma_1\sigma_2(\sigma_1 - \sigma_2)(\sigma_2 - \sigma_3)}{-\sigma_2\sigma_3(\sigma_2 - \sigma_3)^2}$$

$$= 1 - \frac{\sigma_1}{\sigma_2}\left(\frac{\sigma_1 - \sigma_3}{\sigma_2 - \sigma_3}\right) - \frac{\sigma_1}{\sigma_3}\left(\frac{\sigma_1 - \sigma_2}{\sigma_2 - \sigma_3}\right)$$

$$= \cancel{\sigma_2\sigma_3(\sigma_2 - \sigma_3) - \sigma_1\sigma_3}$$

$$\star \quad 1 - \left[\frac{\sigma_1^2 - \sigma_1\sigma_3}{\sigma_2^2 - \sigma_2\sigma_3} + \frac{\sigma_1^2 - \sigma_1\sigma_2}{\sigma_3^2 - \sigma_2\sigma_3}\right]$$

$$= 1 - \left[\frac{\sigma_1^2 - \sigma_1\sigma_3}{\sigma_2(\sigma_2 - \sigma_3)} + \frac{\sigma_1^2 - \sigma_1\sigma_2}{\sigma_3(\sigma_3 - \sigma_2)}\right]$$

$$= 1 - \frac{1}{\sigma_2 - \sigma_3}\left[\frac{\sigma_1^2 - \sigma_1\sigma_3}{\sigma_2} - \frac{(\sigma_1^2 - \sigma_1\sigma_2)}{\sigma_3}\right]$$

$$= 1 - \frac{1}{(\sigma_2 - \sigma_3)\sigma_2\sigma_3}\left[\sigma_1^2\sigma_3 - \sigma_1\sigma_3^2 - \sigma_1^2\sigma_2 + \sigma_1\sigma_2^2\right]$$

$$= 1 - \frac{1}{\sigma_2\sigma_3(\sigma_2 - \sigma_3)}\left[\sigma_1^2(\sigma_3 - \sigma_2) - \sigma_1(\sigma_3^2 - \sigma_2^2)\right]$$

$$= 1 - \frac{1\,(\sigma_3 - \sigma_2)}{\sigma_2\sigma_3(\sigma_2 - \sigma_3)}\left[\sigma_1^2 - \sigma_1(\sigma_3 + \sigma_2)\right]$$

$$= 1 + \frac{1}{\sigma_2\sigma_3}\left[\sigma_1^2 - \sigma_1(\sigma_2 + \sigma_3)\right]$$

$$= \frac{\sigma_2\sigma_3 + \sigma_1^2 - \sigma_1\sigma_2 - \sigma_1\sigma_3}{\sigma_2\sigma_3} = \frac{\sigma_2(\sigma_3 - \sigma_1) - \sigma_1(\sigma_3 - \sigma_1)}{\sigma_2\sigma_3}$$

$$= \frac{(\sigma_1 - \sigma_3)(\sigma_1 - \sigma_2)}{\sigma_2\sigma_3}$$

$$1 - \left(\frac{\tau^2 + \sigma_N^2 - \sigma_3\sigma_N}{\sigma_2^2 - \sigma_2\sigma_3}\right) - \left(\frac{\tau^2 + \sigma_N^2 - \sigma_2\sigma_N}{\sigma_3^2 - \sigma_2\sigma_3}\right)$$

$$= 1 - \left(\frac{\tau^2 + \sigma_N^2 - \sigma_3\sigma_N}{\sigma_2(\sigma_2 - \sigma_3)}\right) - \left(\frac{\tau^2 + \sigma_N^2 - \sigma_2\sigma_N}{\sigma_3(\sigma_3 - \sigma_2)}\right)$$

$$= 1 - \frac{1}{(\sigma_2 - \sigma_3)}\left[\frac{\tau^2 + \sigma_N^2 - \sigma_3\sigma_N}{\sigma_2} + \frac{\tau^2 + \sigma_N^2 - \sigma_2\sigma_N}{\sigma_3}\right]$$

$$= 1 - \frac{1}{(\sigma_2 - \sigma_3)}\left[\frac{(\sigma_3 + \sigma_2)(\tau^2 + \sigma_N^2) - \sigma_3^2\sigma_N - \sigma_2^2\sigma_N}{\sigma_2\sigma_3}\right]$$

$n^2 =$

$$n^2(\sigma_1 - \sigma_3)(\sigma_1 - \sigma_2) = \tau^2 + (\sigma_N - \sigma_2)(\sigma_N - \sigma_3)$$

or

$$n^2 = \frac{\tau^2 + (\sigma_N - \sigma_2)(\sigma_N - \sigma_3)}{(\sigma_1 - \sigma_3)(\sigma_1 - \sigma_2)}$$

We had $\sigma_1 > \sigma_2 > \sigma_3$

for $n^2 \geq 0$

$$\tau^2 + (\sigma_N - \sigma_2)(\sigma_N - \sigma_3) \geq 0$$

If we consider equality.

$$\left[\tau^2 + (\sigma_N - \sigma_2)(\sigma_N - \sigma_3) = 0\right]$$

MOHR'S CIRCLE

$$\tau^2 + \left(\sigma_N - \frac{\sigma_2+\sigma_3}{2}\right)^2 = \left(\frac{\sigma_2-\sigma_3}{2}\right)^2$$

$C: \left(\frac{\sigma_2+\sigma_3}{2}, 0\right)$, $R = \frac{\sigma_2-\sigma_3}{2}$

$C_1: \frac{\sigma_1+\sigma_2}{2}$, $r_1 = \frac{\sigma_1-\sigma_2}{2}$

$C_2: \frac{\sigma_2+\sigma_3}{2}$, $r_2 = \frac{\sigma_2-\sigma_3}{2}$

$C_3: \frac{\sigma_3+\sigma_1}{2}$, $r_3 = \frac{\sigma_3-\sigma_1}{2}$

* Points on the extremeties of the circle represent the principal planes. [$\tau = 0$]

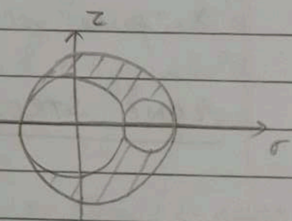
* $\angle$ le b/w the planes is $90°$.

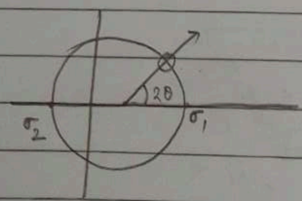
* for a safe design, the point should lie within the circle.

→ On principal planes, shear stress is 0.

→ On planes on which shear is maximum, normal stress is not 0.

(i) if the roots are $\sigma_1, \sigma_2 = \sigma_3$





(ii) if $\sigma_1 = \sigma_2 = \sigma_3$

It becomes a point

→ The stress is equal on all planes.

SPRING

Plane stress and plane strain
↳ small thickness. ⇒ loading and body both should be planar.

Hydrostatic and Deviatoric state of stress:-

$$\boxed{\sigma_{ij} = \underbrace{\sigma_{ij}^{H}}_{\substack{\text{normal} \\ \text{(changes volume)}}} + \underbrace{\sigma_{ij}^{D}}_{\substack{\text{shear} \\ \text{(changes shape)}}}}$$

defined w.r.t. the principal axes $\left[\sigma_{ij}^{D} = \sigma_{ij} - \sigma_{ij}^{H} = \sigma_{ij} - \frac{\sigma_{ii}}{3}\right]$

⇒ Octahedral planes. (8 such planes)

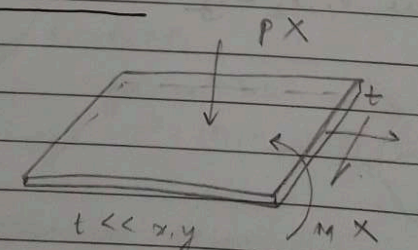
STATE OF PURE SHEAR -

$$I_1 = \sigma_x + \sigma_y + \sigma_z = 0$$

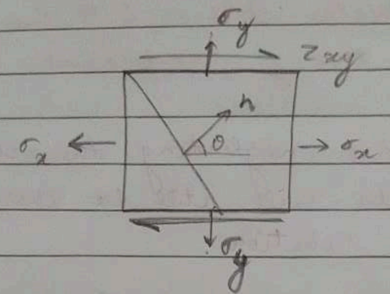
First invariant is zero and octahedral planes are free from normal stress.

* curve fitting for stress - strain plot
↳ "pure" shear or "pure" hydrostatic stress

PLANE STRESS -



$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



$$[a_{ij}] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$T_{ij} = \sigma_{ij} n_j$$

$$T = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} n\cos\theta \\ n\sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{xx}\cos\theta + \sigma_{xy}\sin\theta \\ \sigma_{xy}\cos\theta + \sigma_{yy}\sin\theta \end{bmatrix} \quad [\|n\| = 1]$$

$$\sigma_N = \vec{T} \cdot \underbrace{\vec{n}}_{[a_{ij}]}$$

$$= \sigma_{xx}\cos^2\theta + \sigma_{xy}\sin\theta\cos\theta$$
$$\sigma_{xy}\sin\theta\cos\theta + \sigma_{yy}\sin^2\theta$$

$$= \sigma_{xx}\left(\frac{\cos 2\theta + 1}{2}\right) + \frac{\sigma_{xy}}{2}\sin 2\theta$$

$$+ \frac{\sigma_{xy}}{2}\sin 2\theta + \sigma_{yy}\left(\frac{1 - \cos 2\theta}{2}\right)$$

$$= \frac{\sigma_{xx} + \sigma_{yy}}{2} + \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)\cos 2\theta$$

$$+ \sigma_{xy} \cancel{\theta} \sin 2\theta .$$

P.T.O.

$\cos 2\theta$
$= 2\cos^2\theta - 1$
$= 1 - 2\sin^2\theta$

$\sigma_1 > \sigma_2 > \sigma_3$ still holds.

There are 3 circles still, but we are not accounting for the z component here, simply neglecting it.

→ Although we are neglecting the stress, the strain components will still be there because of the Poisson's relation.

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2}$$

$$\tau = \frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

for 0 shear stress

$$\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta = 0$$

or, $$\boxed{\tan 2\theta = \frac{-2\tau_{xy}}{\sigma_x - \sigma_y}}$$

for max. shear stress

$$\frac{\partial \tau}{\partial \theta} = 0$$

$$\frac{\sigma_x - \sigma_y}{2} \cos 2\theta . 2 - \tau_{xy} \sin 2\theta . 2 = 0$$

or, $$\boxed{\tan 2\theta = \frac{\sigma_x - \sigma_y}{2\tau_{xy}}}$$

→ Necking angle. $= 45°$

(depends on crystal structure of material)

$\begin{array}{r} 4225 \\ 625 \\ \hline 4850 \end{array}$ $\quad \begin{array}{r} 69.6 \\ -15.0 \\ \hline 54.6 \end{array}$ $\quad \begin{array}{r} 69.6 \\ +15 \\ \hline 84.6 \end{array}$

Stick to one convention

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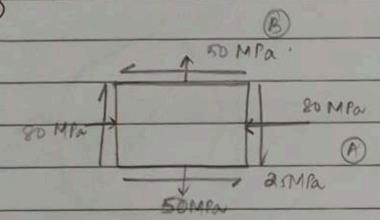
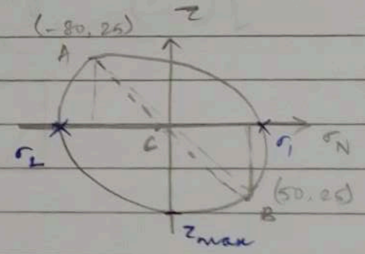
Mohr's circle for plane stress :-

* Normal stress (tensile) is positive
* τ (+ve) downwards (a/c/w sense)

$\theta \longrightarrow 2\theta$

Ques: Draw the Mohr's circle and then determine principal stresses and maximum shear stresses for the given state of stress.





$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$\boxed{C \pm R}$

$= -15 \pm \sqrt{4225 + 625}$

$= -15 \pm 69.6 = -84.6, \; 54.6$
$\quad (\sigma_2) \quad (\sigma_1)$

$$\sigma_{ij} = \begin{bmatrix} -80 & 25 \\ 25 & 50 \end{bmatrix} \qquad I_1 = -30$$

$$\sigma_{ij}^{P} = \begin{bmatrix} 54.6 & 0 \\ 0 & -84.6 \end{bmatrix} \qquad I_1 = -30$$

$\Rightarrow$ correct

$\sigma_1 = 54.6, \; \sigma_2 = 0, \; \sigma_3 = -84.6$

τ_{max}

$45^\circ \rightarrow 90^\circ$ (2θ from θ)

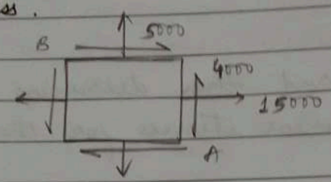
coordinates

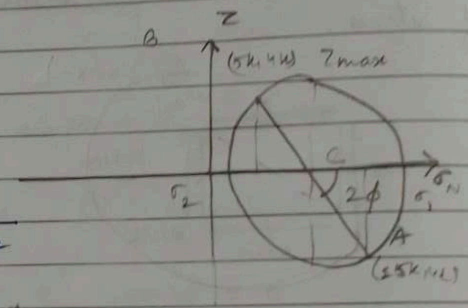
tangent

Ques: Using Mohr's circle, determine the following quantities:

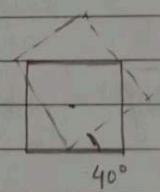
(a) the stresses on an element inclined at 40°.

(b) the principal stresses and maximum shear stress.





$$\sigma_{1,2} = \left(\frac{\sigma_x - \sigma_y}{2}\right) \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$



* BOUNDARY CONDITIONS / STRESS FIELD

$$\sigma_{ij} = \begin{bmatrix} (4 - x_1^2)x_2 + \frac{2}{3}x_2^3 & -(4 - x_2^2)x_1 & 0 \\ -(4 - x_2^2)x_1 & -\frac{1}{3}(x_2^3 - 12x_2) & 0 \\ 0 & 0 & (3 - x_1^2)x_2 \end{bmatrix}$$

Check $\frac{\partial \sigma_{ij}}{\partial x_j} = 0$

$\Rightarrow -2x_1x_2 + 2x_1x_2 + 0 = 0$

$\cancel{-2x_1x_2}\; -(4 - x_2^2) - \frac{1}{3}(3x_2^2 - 4) + 0 = 0$

* Everything flows — Spherical domes — shear thickening
* Visco-elastic substances

→ Traction-free surface does not imply all stress compon[ents] are zero.
[DIFFERENT FROM STRESS-FREE]